Lecture 4 01/29/2018

Review of Electrostatics (Cont'd)

Separation of variables

This is a method of solving the Laplace equation (as well as other homogeneous linear partial differential equations). It depends on the separability of Laplacian along the coordinate directions of a coordinate system.

Cartesian Coordinates

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(x,y,z) = X(x)\,Y(y)\,Z(z) \Rightarrow YZ\frac{d^2X}{dx^2} + ZX\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0 \Rightarrow$$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x \text{ only}} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{function of } y \text{ only}} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\text{function of } z \text{ only}} = 0$$

The only possibility for the equality to hold is that each of the three terms on the left-hand side is a constant:

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\alpha^2,\quad \frac{1}{Y}\frac{d^2Y}{dy^2}=-\beta^2,\quad \frac{1}{Z}\frac{d^2Z}{dz^2}=\alpha^2+\beta^2$$

We then find three ordinary differential equations:

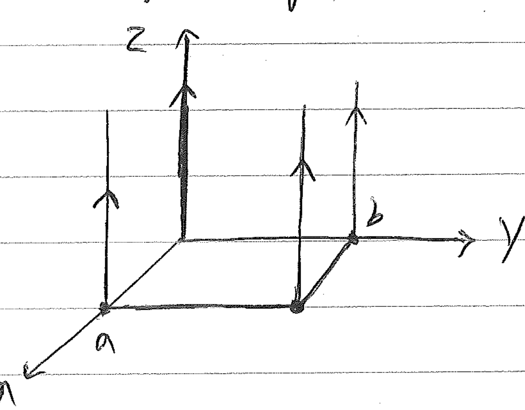
$$\left\{\begin{aligned}&\frac{d^2X}{dx^2}+\alpha^2X=0\\&\frac{d^2Y}{dy^2}+\beta^2Y=0\\&\frac{d^2Z}{dz^2}-(\alpha^2+\beta^2)Z=0\end{aligned}\right.\quad\Rightarrow\quad \boxed{X\propto e^{\pm i\alpha x},\ Y\propto e^{\pm i\beta y},\ Z\propto e^{\pm\sqrt{\alpha^2+\beta^2}\,z}}$$

We note that α and β are completely arbitrary at this point and they can be any complex number. Also, any linear combination of the products of these solutions will be a solution to the Laplace equation.

Example: A rectangular box with open top.

$\Phi=0$ at $x=0,\ x=a,\ y=0,\ y=b$

$\Phi(x,y,0)=f(x,y)$



For this problem, periodicity of the boundary condition with respect to x and y implies that α, β are real. Also, because $z = \infty$ is included, Z must be an exponentially decaying function of z.

Therefore:

$$\Phi(x,y,z) = \sum_{\alpha,\beta} (A_\alpha \sin\alpha x + B_\alpha \cos\alpha x)(C_\beta \sin\beta y + D_\beta \cos\beta y)\, e^{-\sqrt{\alpha^2+\beta^2}\, z}$$

$$\Phi(x=0) = 0 \Rightarrow B_\alpha = 0$$

$$\Phi(y=0) = 0 \Rightarrow D_\beta = 0$$

$$\Phi(x=a) = 0 \Rightarrow \alpha_n = \frac{n\pi}{a} \qquad n = 1, 2, \ldots$$

$$\Phi(y=b) = 0 \Rightarrow \beta_m = \frac{m\pi}{b} \qquad m = 1, 2, \ldots$$

This results in:

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} E_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{-\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\, \pi z}$$

Imposing the boundary condition on the xy plane, we find:

$$f(x,y) = \sum_{n,m} E_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \Rightarrow$$

$$\boxed{E_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dx\, dy}$$

In general, when more than one inhomogeneous boundary condition is given, the solution can be written as a superposition of solutions obtained for all but one boundary held at zero potential.

Polar Coordinates

$$\nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho \frac{\partial \Phi}{\partial \rho}\right] + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0 \qquad \text{(no dependence on } z\text{)}$$

$$\Phi(\rho,\phi) = R(\rho)\Psi(\phi) \Rightarrow \Psi \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{R}{\rho^2}\frac{d^2\Psi}{d\phi^2} = 0 \Rightarrow$$

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = 0$$

$$\begin{cases} \dfrac{1}{\Psi}\dfrac{d^2\Psi}{d\phi^2} = -\nu^2 \\[2ex] \dfrac{\rho}{R}\dfrac{d}{d\rho}\left(\rho \dfrac{dR}{d\rho}\right) = \nu^2 \end{cases} \Rightarrow \begin{cases} \dfrac{d^2\Psi}{d\phi^2} + \nu^2 \Psi = 0 \\[2ex] \rho \dfrac{d}{d\rho}\left(\rho \dfrac{dR}{d\rho}\right) - \nu^2 R = 0 \end{cases}$$

$$\frac{d^2\Psi}{d\phi^2} + \nu^2 \Psi = 0 \Rightarrow \Psi \propto e^{\pm i\nu\phi}$$

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) - \nu^2 R = 0 \Rightarrow \rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} - \nu^2 R = 0$$

Taking $R \propto \rho^\lambda$, we find:

$$\lambda(\lambda-1)\rho^\lambda + \lambda\rho^\lambda - \nu^2\rho^\lambda = 0 \Rightarrow \lambda = \pm\nu$$

Therefore, the general solution is:

$$\boxed{\Phi(\rho,\phi) = \sum_{\nu\neq 0}\left(A_\nu \rho^\nu + B_\nu \rho^{-\nu}\right)\left(C_\nu e^{i\nu\phi} + D_\nu e^{-i\nu\phi}\right) + (A + B\ln\rho)}$$

For $\nu = 0$, the differential equation for R has a solution given as $A + B\ln\rho$. The only solution of Ψ that is physically acceptable for $\nu = 0$ is a constant.

In problems where the full range $\phi = (0, 2\pi)$ is involved, Φ must be periodic with respect to ϕ. This implies that:

$$e^{\pm i\nu 2\pi} = 1 \Rightarrow \nu = m \qquad (m \text{ a non-zero integer})$$

Thus, in this case:

$$\boxed{\Phi(\rho,\phi) = (A + B\ln\rho) + \sum_{m\neq 0}\left(A_m \rho^m + B_m \rho^{-m}\right)e^{im\phi}}$$

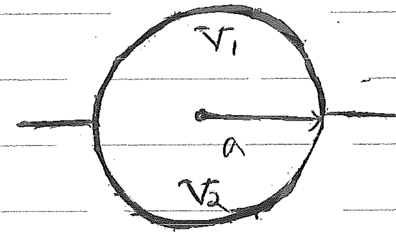
For problem where inside a volume is of interest, negative powers of ρ are not permitted (hence excluding ρ^{-m} terms for $m>0$, and ρ^{m} terms for $m<0$). Similarly, $\ln \rho$ term will not be permitted.

In this case, the solution has the following form:

$$\Phi(\rho,\phi) = \sum_{m=-\infty}^{\infty} A_m \rho^{|m|} e^{im\phi}$$

Example: An infinite hollow cylinder with potential V_1 on one half and potential V_2 on the other half.

$$\Phi = \begin{cases} V_1 & \rho = a,\ 0<\phi<\pi \\ V_2 & \rho = a,\ \pi<\phi<2\pi \end{cases}$$



Starting from:

$$\Phi(\rho,\phi) = \sum_{m=-\infty}^{+\infty} A_m \rho^{|m|} e^{im\phi}$$

We find:

$$A_m |a|^m = \frac{1}{2\pi}\left[\int_0^{\pi} V_1 e^{-im\phi}\, d\phi + \int_{\pi}^{2\pi} V_2 e^{-im\phi}\, d\phi\right] \Rightarrow$$

$$\boxed{A_m = \frac{1}{|a|^m}\frac{1}{2\pi}\left[\int_0^{\pi} V_1 e^{-im\phi}\, d\phi + \int_{\pi}^{2\pi} V_2 e^{-im\phi}\, d\phi\right]}$$

In the special case that $V_1 = V_2 = V$ (i.e., Constant potential on the entire cylinder), we find:

$$A_m = V_0 \, \delta_{m,0}$$

This implies that $\Phi = V$ everywhere inside the cylinder. The same situation holds if volume of interest is outside the cylinder.